Shore

Year 12
Term II Examination
24 April 2015

Mathematics

Examination Number:

Set:

General Instructions

- Reading time − 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Answer Questions 1–10 on the Multiple Choice Answer Sheet provided
- In Questions 11–16, show relevant mathematical reasoning and/or calculations
- Start each of Questions 11−16 in a new writing booklet
- Write your examination number on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a blank booklet marked with your examination number and “N/A” on the front cover

Total marks − 100

Section I Pages 3−6

10 marks

- Attempt Questions 1−10
- Allow about 15 minutes for this section

Section II Pages 7−13

90 marks

- Attempt Questions 11−16
- Allow about 2 hours and 45 minutes for this section

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Section I

10 marks
Attempt Questions 1−10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1−10.

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1 What is the value of $\dfrac{\sqrt{3.84}}{2.65+7.7}$ correct to two decimal places?

(A) 0.19

(B) 0.61

(C) 5.28

(D) 8.44

2 What are the conditions for the expression ax^2+bx+c to be positive definite?

(A) $a>0$ and $\Delta>0$

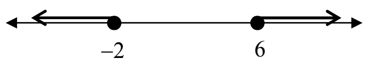
(B) $c>0$ and $\Delta>0$

(C) $a>0$ and $\Delta<0$

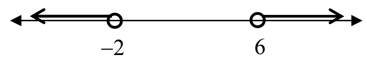
(D) $c>0$ and $\Delta<0$

3 Which of the following graphs represents the solution to $|x-2|>4$?

(A) 

(B) 

(C) 

(D) 

4 The curve $y = f(x)$ is decreasing and concave down.

Which one of the following applies to this curve?

(A) $f'(x) > 0$ and $f''(x) > 0$

(B) $f'(x) > 0$ and $f''(x) < 0$

(C) $f'(x) < 0$ and $f''(x) > 0$

(D) $f'(x) < 0$ and $f''(x) < 0$

5 A parabola has equation $x^2 = 8(y+2)$.

What are the coordinates of its vertex (V) and focus (F) respectively?

(A) $V(0,2)$ and $F(0,0)$

(B) $V(0,-2)$ and $F(0,0)$

(C) $V(0,2)$ and $F(0,-4)$

(D) $V(0,-2)$ and $F(0,-4)$

6 What is an equivalent expression for $4^x + 4^x + 4^x + 4^x$?

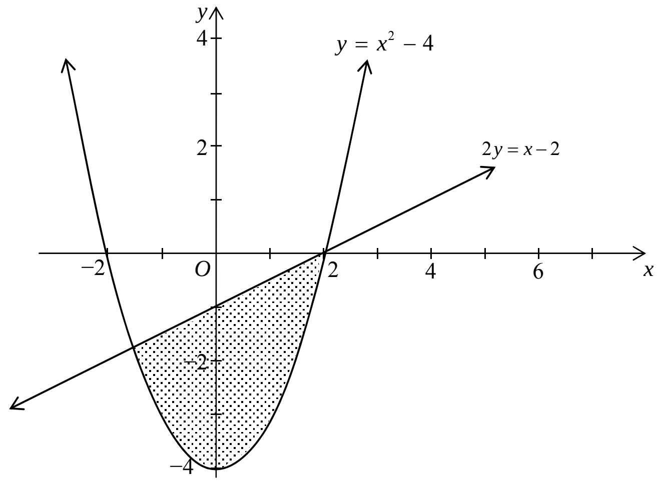
(A) 4^{4x}

(B) 16^{4x}

(C) 16^x

(D) 4^{x+1}

7 The diagram shows the region enclosed by $y = x^2 - 4$ and $2y = x - 2$.



Which of the following pairs of inequalities describes the shaded region in the diagram?

(A) $y \le x^2 - 4$ and $2y \le x - 2$

(B) $y \le x^2 - 4$ and $2y \ge x - 2$

(C) $y \ge x^2 - 4$ and $2y \le x - 2$

(D) $y \ge x^2 - 4$ and $2y \ge x - 2$

8 Which of the following statements is true for the geometric sequence 24, 12, 6,……..?

(A) The fourth term is 0.

(B) The sum of the first four terms is 44.

(C) The sum of the series will never exceed 48.

(D) There are an infinite number of negative terms.

9 If $\sin\theta = -\frac{3}{5}$ and $\cos\theta < 0$, what is the value of $\tan\theta$?

(A) $\frac{3}{4}$

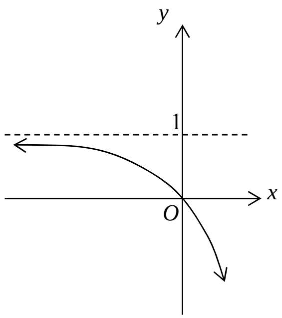
(B) $\frac{4}{3}$

(C) $-\frac{3}{4}$

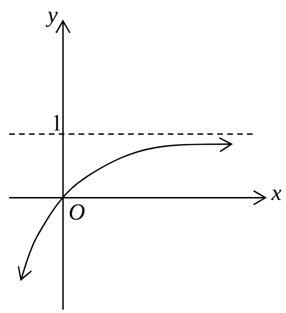
(D) $-\frac{4}{3}$

10 Which of the following graphs could have equation $y = 1 - 2^{-x}$?

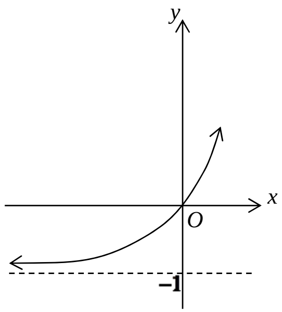
(A)



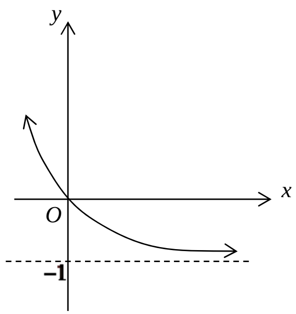
(B)



(C)



(D)



End of Section I

Section II

90 marks
Attempt Questions 11−16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) Solve $\frac{2x-3}{8} - \frac{x-5}{6} = 1$. **2**

(b) Fully factorise $x^3 - x^2 - 4x + 4$. **2**

(c) Express $\frac{3-\sqrt{5}}{3+\sqrt{5}}$ as a simplified fraction with a rational denominator. **2**

(d) Evaluate $\lim_{x\to 3} \frac{x-3}{x^2-9}$. **2**

(e) How many sides does a regular polygon have if each interior angle is 168° ? **2**

(f) Evaluate $\sum_{k=2}^{5} (-1)^k \frac{1}{k}$. **2**

(g) The roots of the quadratic equation $3x^2 + 6x - 2 = 0$ are α and β .

(i) Find the value of $\alpha + \beta$. **1**

(ii) Find the value of $\alpha^3\beta^2 + \alpha^2\beta^3$. **2**

Question 12 (15 marks) Use a SEPARATE Writing Booklet

(a) Differentiate the following with respect to x.

(i) $\dfrac{1}{2\sqrt{x}}$ **2**

(ii) xe^{x^2} **2**

(b) (i) Find $\displaystyle\int \frac{1}{(4x-1)^4}\,dx$. **2**

(ii) Find $\displaystyle\int \sqrt{e^x}\,dx$. **2**

(c) Solve the inequality $2x^2 - 3x - 2 \geq 0$. **2**

(d) Find the gradient, in simplest form, of the tangent to the curve $y = e^{3x}$ at the point where $x = \log_e 2$. **2**

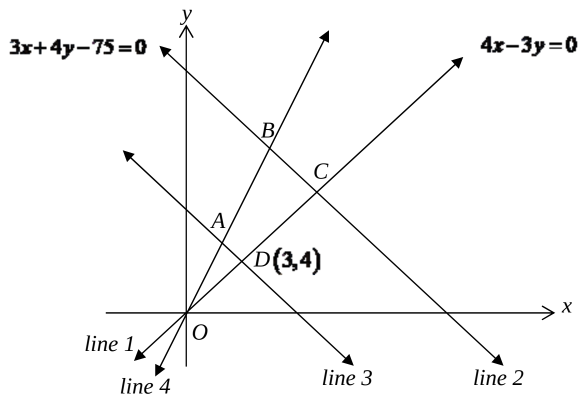
(e) Simplify $\dfrac{2\cos^2 x - 2}{2\sin x \cos x}$. **3**

Question 13 (15 marks) Use a SEPARATE Writing Booklet

(a) Solve $2\sin^2\theta - 1 = 0$ for $0° \leq \theta \leq 360°$. **3**

(b) Find the area between the curve $y = x^2 - 2x$, the x axis and the lines $x = 1$ and $x = 3$. **3**

(c) In the diagram below, *line 1* has the equation $4x - 3y = 0$, *line 2* has equation $3x + 4y - 75 = 0$, *line 3* intersects with *line 1* at the point $D(3,4)$ and *line 4* passes through the origin, O.



NOT TO SCALE

(i) Show that *line 1* and *line 2* are perpendicular. **2**

(ii) Determine the equation of *line 3*, which passes through point $D(3,4)$ and is parallel to *line 2*. **2**

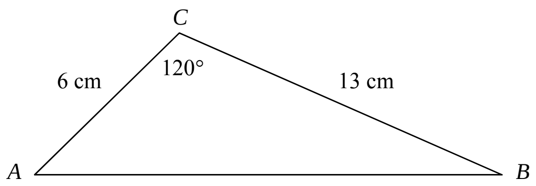
(iii) Show that the perpendicular distances from the origin, O, to *line 2* and *line 3* are 15 units and 5 units respectively. **2**

(iv) *Line 4* intersects *lines 2* and *3* at points B and A respectively. Determine the ratio $OB : OA$. **1**

(v) C is the point of intersection of *lines 1* and *2*. Show that ΔOBC and ΔOAD are similar. **2**

Question 14 (15 marks) Use a SEPARATE Writing Booklet

(a)



NOT TO SCALE

(i) Find the length of side AB correct to 2 significant figures. **2**

(ii) Find the exact area of ΔABC. **2**

(b) Consider the quadratic equation in x, $x^2 - (k+2)x + 2k = 0$.

(i) Find the value(s) of k if the roots are reciprocals of each other. **2**

(ii) Show that the roots are always real for all values of k. **3**

(c) Use Simpsons rule with 5 function values to find an approximation to **2**

$\int_3^5 \ln(x-2)\, dx$. Write your answer correct to 2 decimal places.

(d) Brendan trained for the City to Surf by running each day for 14 consecutive days. Each day he ran 500 metres further than the previous day.

(i) On the final day Brendan ran 16 km. How far did he run on the first day? **2**

(ii) Find the total distance Brendan ran in the 14 days. **2**

Question 15 (15 marks) Use a SEPARATE Writing Booklet

(a) Consider the series $1 + (1-x) + (1-x)^2 + (1-x)^3 + \ldots\ldots$ **2**

For what values of x will this series have a limiting sum?

(b) Consider the curve $f(x) = x^4 - 8x^2 + 16$.

(i) Prove that the function is even. **1**

(ii) Show that $f'(x) = 4x(x-2)(x+2)$. **2**

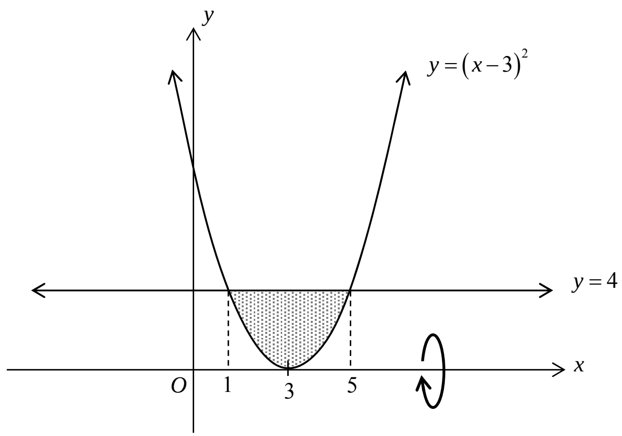
(iii) Find the stationary points and determine their nature. **3**

(iv) Sketch the curve showing all important features. **2**

(v) Find the values of x for which the curve is increasing. **1**

(c) The area between the curve $y = (x-3)^2$ and the line $y = 4$ is rotated about the x axis as indicated in the diagram. **4**

Find the exact volume of the solid of revolution.



Question 16 (15 marks) Use a SEPARATE Writing Booklet

(a) (i) Simplify $\log_a b^2 \times \log_b a$. **2**

(ii) Solve $\log_2(x-2)+\log_2(x+2)=5$ for $x>2$. **2**

(b) Bob borrows \$30 000 from the bank to buy a new car. The loan plus interest and charges are to be repaid at the end of each month in equal monthly instalments of *\$B* over 5 years. Interest is charged at 6% per annum and is charged on the balance owing at the beginning of each month. Furthermore, a bank charge of \$20 is added to the account balance at the end of each month.

Let A_n be the amount owing at the end of *n* months.

(i) Write down an expression for A_1 . **1**

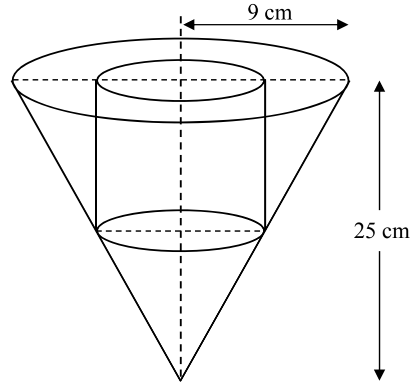
(ii) Show that the amount owing after 2 months is given by **2**

$$A_2 = \$30\,000 \times 1.005^2 - (B-20)(1+1.005)$$

(iii) Find Bob's monthly instalment, *\$B*, correct to the nearest cent. **3**

Question 16 continues on page 13

(c) A cylinder is inscribed inside a cone of radius 9 cm and height 25 cm.



(i) Use similar triangles to show that the height *h* of the cylinder is given by **1**

$$h = \frac{25(9-r)}{9}$$

where *r* is the radius of the cylinder.

(ii) Show that the volume *V* of the cylinder is given by **1**

$$V = \frac{25\pi}{9}\left(9r^2 - r^3\right).$$

(iii) Hence find the maximum possible volume of the cylinder. **3**

End of paper

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2U Y12 Mid-Year Exam Solutions 2015

(1) $0.1893\ldots \doteq 0.19$ (A)

(2) $a > 0,\ \Delta < 0$ (C)

(3) $|x-2| > 4$

$x-2 < -4$ or $x-2 > 4$

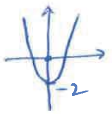
$x < -2$ or $x > 6$

(D)

(4) $f'(x) < 0$

$f''(x) < 0$ (D)

(5)



$a = 2$

$V(0,-2)$

$F(0,0)$ (B)

(6) $4^x + 4^x + 4^x + 4^x = 4 \times 4^x$

$= 4^{x+1}$ (D)

(7) $y \geq x^2 - 4$

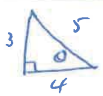
$2y \leq x - 2$ (C)

(8) $24, 12, 6, \ldots$

$S_\infty = \dfrac{24}{1-\frac{1}{2}}$

$= 48$ (C)

(9)



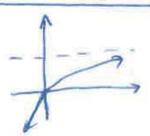
$\sin\theta < 0$

$\cos\theta < 0$

$\tan\theta > 0$

$\tan\theta = \frac{3}{4}$ (A)

(10)



$y = 1 - 2^{-x}$ (B)

Question 11

(a) $\dfrac{2x-3}{8} - \dfrac{x-5}{6} = 1$

$3(2x-3) - 4(x-5) = 24$

$6x - 9 - 4x + 20 = 24$ [2]

$2x + 11 = 24$

$2x = 13$

$x = 6\frac{1}{2}$

(b) $x^3 - x^2 - 4x + 4$

$= x^2(x-1) - 4(x-1)$

$= (x^2-4)(x-1)$

$= (x-2)(x+2)(x-1)$ [2]

(c) $\dfrac{3-\sqrt{5}}{3+\sqrt{5}} \times \dfrac{3-\sqrt{5}}{3-\sqrt{5}}$

$= \dfrac{9 - 6\sqrt{5} + 5}{9-5}$

$= \dfrac{14 - 6\sqrt{5}}{4}$

$= \dfrac{7 - 3\sqrt{5}}{2}$ [2]

(d) $\displaystyle\lim_{x\to 3} \frac{x-3}{x^2-9}$

$= \displaystyle\lim_{x\to 3} \frac{x-3}{(x-3)(x+3)}$

$= \displaystyle\lim_{x\to 3} \frac{1}{x+3}$ [2]

$= \dfrac{1}{6}$

(e) Each Ext. $\angle = 180° - 168°$

$= 12°$

No. of sides $= \dfrac{360}{12}$

$= 30$ [2]

(f) $\displaystyle\sum_{k=2}^{5} (-1)^k \cdot \frac{1}{k}$

$= (-1)^2 \cdot \frac{1}{2} + (-1)^3 \cdot \frac{1}{3} + (-1)^4 \cdot \frac{1}{4} + (-1)^5 \cdot \frac{1}{5}$

$= \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5}$

$= \dfrac{13}{60}$ [2]

(g) $3x^2 + 6x - 2 = 0$

(i) $\alpha + \beta = \dfrac{-6}{3}$

$= -2$ [1]

(ii) $\alpha^3\beta^2 + \alpha^2\beta^3$

$= \alpha^2\beta^2(\alpha+\beta)$

$= (\alpha\beta)^2(\alpha+\beta)$

$= \left(-\frac{2}{3}\right)^2 \times -2$ [2]

$= -\dfrac{8}{9}$

Question 12

(a) (i) $y = \dfrac{1}{2\sqrt{x}}$

$= \frac{1}{2} x^{-\frac{1}{2}}$

$\dfrac{dy}{dx} = -\frac{1}{4} x^{-\frac{3}{2}}$ [2]

$\left(\text{or } -\dfrac{1}{4\sqrt{x^3}}\right)$

(ii) $y = x e^{x^2}$

$y' = vu' + uv'$

$= e^{x^2} \times 1 + x \cdot 2x e^{x^2}$

$= e^{x^2}(1 + 2x^2)$ [2]

(b) (i) $\displaystyle\int (4x-1)^{-4}\, dx$

$= \dfrac{(4x-1)^{-3}}{-3 \times 4} + c$

$= \dfrac{(4x-1)^{-3}}{-12} + c$

$= -\dfrac{1}{12(4x-1)^3} + c$ [2]

(ii) $\displaystyle\int \sqrt{e^x}\, dx$

$= \displaystyle\int (e^x)^{\frac{1}{2}}\, dx$

$= \displaystyle\int e^{\frac{x}{2}}\, dx$ [2]

$= 2e^{\frac{x}{2}} + c$

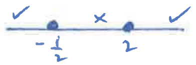
$\left(\text{or } 2\sqrt{e^x} + c\right)$

Question 12 continued

(c) $2x^2 - 3x - 2 \geq 0$ [2]

$(2x+1)(x-2) \geq 0$

Critical Points: $x = -\frac{1}{2}$ $x = 2$



$x \leq -\frac{1}{2}$ or $x \geq 2$

(d) $y = e^{3x}$

$\frac{dy}{dx} = 3e^{3x}$

at $x = \ln 2$ $m_T = 3e^{3\ln 2}$

$= 3e^{\ln 8}$

$= 3 \times 8$

$= 24$ [2]

(e) $\frac{2\cos^2 x - 2}{2\sin x \cos x}$

$= \frac{2(\cos^2 x - 1)}{2\sin x \cos x}$

$= \frac{-\sin^2 x}{\sin x \cos x}$

$= -\frac{\sin x}{\cos x}$

$= -\tan x$ [3]

Question 13

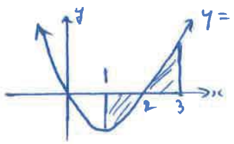
(a) $2\sin^2\theta - 1 = 0$

$\sin^2\theta = \frac{1}{2}$

$\sin\theta = \pm\frac{1}{\sqrt{2}}$ [3]

$\theta = 45°, 135°, 225°, 315°$

(b)



$A = \left|\int_1^2 (x^2 - 2x)\,dx\right| + \int_2^3 (x^2 - 2x)\,dx$

$= \left|\left[\frac{x^3}{3} - x^2\right]_1^2\right| + \left[\frac{x^3}{3} - x^2\right]_2^3$

$= \left|\left(\frac{8}{3} - 4\right) - \left(\frac{1}{3} - 1\right)\right| + \left(\frac{27}{3} - 9\right) - \left(\frac{8}{3} - 4\right)$

$= \left|-\frac{4}{3} + \frac{2}{3}\right| + 0 + \frac{4}{3}$

$= \frac{2}{3} + \frac{4}{3}$

$= 2$ units^2 [3]

(c) (i) $4x - 3y = 0 \Rightarrow m_1 = \frac{4}{3}$

$3x + 4y - 75 = 0 \Rightarrow m_2 = -\frac{3}{4}$

$m_1 \times m_2 = \frac{4}{3} \times -\frac{3}{4}$ [2]

$= -1$ $\therefore$ line 1 ⊥ line 2

(ii) $m_2 = -\frac{3}{4} \Rightarrow m = -\frac{3}{4}$ $D(3, 4)$

Eq'n is: $y - 4 = -\frac{3}{4}(x - 3)$

$4y - 16 = -3x + 9$

$3x + 4y - 25 = 0$ [2]

Question 13 continued

(iii) $d_2 = \frac{|3(0) + 4(0) - 75|}{\sqrt{3^2 + 4^2}}$

$= \frac{|-75|}{5}$

$= 15$ units

$d_3 = \frac{|3(0) + 4(0) - 25|}{\sqrt{3^2 + 4^2}}$

$= \frac{|-25|}{5}$

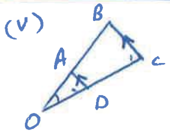
$= 5$ units [2]

(iv) $OB : OA = OC : OD$

$= 15 : 5$

$= 3 : 1$ [1]

(v)



$\angle O$ is common

$\angle BCO = \angle ADO$

corresponding $\angle$'s, $AD \parallel BC$

$\therefore \triangle OBC \,|||\, \triangle OAD$

equiangular [2]

[OR sides in proportion + included $\angle$'s equal]

Question 14

(a) (i) $AB^2 = 6^2 + 13^2 - 2 \times 6 \times 13 \cos 120°$

$= 283$ [2]

$AB = 16.8226\ldots$

$= 17$ cm (to 2 s.f.)

(ii) Area $= \frac{1}{2} \times 6 \times 13 \sin 120°$

$= 39 \times \frac{\sqrt{3}}{2}$ [2]

$= \frac{39\sqrt{3}}{2}$ cm^2

(b) $x^2 - (k+2)x + 2k = 0$

(i) let roots be α and $\frac{1}{\alpha}$

$\alpha \times \frac{1}{\alpha} = 2k$

$1 = 2k$ [2]

$k = \frac{1}{2}$

(ii) For real roots $\Delta \geq 0$

$\Delta = (k+2)^2 - 4 \times 1 \times 2k$

$= k^2 + 4k + 4 - 8k$

$= k^2 - 4k + 4$ [3]

$= (k-2)^2$

≥ 0 for all values of k

(c) $\int_3^5 \ln(x-2)\,dx$

x	3	3.5	4	4.5	5
y	y_F	y_1	y_2	y_3	y_L

$\doteqdot \frac{h}{3}\left[y_F + y_L + 4(\text{odd}) + 2(\text{even})\right]$

$= \frac{0.5}{3}\left[\ln 1 + \ln 3 + 4(\ln 1.5 + \ln 2.5) + 2(\ln 2)\right]$

$= 1.295\ldots$ [2]

$= 1.30$ (to 2 d.p.)

(d) $d = 0.5$ km, $n = 14$

(i) $T_{14} = 16$

$a + 13 \times 0.5 = 16$ [2]

$a = 9.5$

Brendan ran 9.5 km on first day

(ii) $S_{14} = \frac{14}{2}(9.5 + 16)$

Total $= 178.5$ km [2]

Question 15

(a) $-1 < 1-x < 1$

$-2 < -x < 0$

$2 > x > 0$ [2]

$0 < x < 2$

(b) (i) $f(x) = x^4 - 8x^2 + 16$

$f(-x) = (-x)^4 - 8(-x)^2 + 16$

$= x^4 - 8x^2 + 16$

$= f(x)$ [1]

$\therefore f(x)$ is even

(ii) $f'(x) = 4x^3 - 16x$

$= 4x(x^2 - 4)$ [2]

$= 4x(x-2)(x+2)$

(iii) For S.P's $4x(x-2)(x+2) = 0$

$x=0,\ x=2,\ x=-2$

at $x=0$, $y=16$

x	-1	0	1
y'	12	0	-12
	/	—	\

at $x=2$, $y = 16-32+16 = 0$

x	1	2	3
y'	-12	0	60
	\	—	/

at $x=-2$, $y = 16-32+16 = 0$

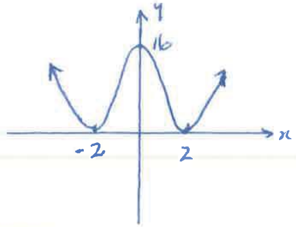
x	-3	-2	-1
y'	-60	0	12
	\	—	/

$\therefore$ Minimum S.P's at $(-2,0)$ & $(2,0)$

Maximum S.P. at $(0,16)$

[3]

(iv)



[2]

(v) Increasing for $-2 < x < 0$, $x > 2$

[1]

(c)

$$V = \pi\int_1^5 4^2\,dx - \pi\int_1^5 \left((x-3)^2\right)^2 dx$$

$$= \pi\int_1^5 16 - (x-3)^4\,dx$$

$$= \pi\left[16x - \frac{(x-3)^5}{5}\right]_1^5$$

$$= \pi\left[\left(80 - \frac{32}{5}\right) - \left(16 - \frac{-32}{5}\right)\right]$$

$$= \pi\left[\frac{368}{5} - \frac{112}{5}\right]$$

$$= \frac{256\pi}{5} \text{ units}^3$$

[4]

Question 16

(a) (i) $\log_a b^2 \times \log_b a$

$= 2\log_a b \times \log_b a$

$= 2\dfrac{\log b}{\log a} \times \dfrac{\log a}{\log b}$ [2]

$= 2$

(ii) $\log_2(x-2) + \log_2(x+2) = 5$

$\log_2 (x-2)(x+2) = 5$

$(x-2)(x+2) = 2^5$

$x^2 - 4 = 32$ [2]

$x^2 = 36$

$x = 6 \quad (x > 2)$

(b) (i) $A_1 = 30000(1.005) - B + 20$

[1]

(ii) $A_2 = A_1(1.005) - B + 20$

$= [30000(1.005) - (B-20)]1.005 - (B-20)$

$= 30000(1.005)^2 - (B-20)1.005 - (B-20)$

$= 30000(1.005)^2 - (B-20)(1+1.005)$

[2]

(iii) By observation:

$A_{60} = 30000(1.005)^{60} - (B-20)(1 + 1.005 + 1.005^2 + \ldots + 1.005^{59})$

$A_{60} = 0$, since paid off in 5 years

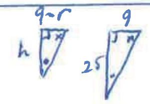
$0 = 30000(1.005)^{60} - (B-20) \times \dfrac{1(1.005^{60} - 1)}{1.005 - 1}$

$\dfrac{(B-20)(1.005^{60} - 1)}{0.005} = 30000(1.005)^{60}$

$B = \dfrac{30000(1.005)^{60}}{1.005^{60} - 1} \times 0.005 + 20$

$= \$599.98$ [3]

(c) (i) $\dfrac{h}{25} = \dfrac{9-r}{9}$



$h = \dfrac{25(9-r)}{9}$ [1]

(ii) $V = \pi r^2 h$

$= \pi r^2 \times \dfrac{25(9-r)}{9}$

$= \dfrac{25\pi}{9}(9r^2 - r^3)$ [1]

(iii) $\dfrac{dV}{dr} = \dfrac{25\pi}{9}(18r - 3r^2)$

For max. $\dfrac{25\pi}{9}(18r - 3r^2) = 0$

~~$r=0$~~, $18 - 3r = 0$, $r = 6$ [3]

when $r=6$ $\dfrac{d^2V}{dr^2} = \dfrac{25\pi}{9}(18 - 6r)$

$= \dfrac{25\pi}{9}(18 - 36) < 0$

$\therefore$ maximum

Maximum $V = \dfrac{25\pi}{9}(9 \times 6^2 - 6^3)$

$= 300\pi \text{ cm}^3$.